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Modeling of Thermoelastic Hollow Cylinder by the Application of Fractional Order Theory

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Abstract

The present work deals with the Mathematical Modelling of Thermoelastic Hollow Cylinder occupying the space $D:\ a \le r \le b;\ 0 \le z \le h$ *by the Application of Fractional Order Theory. A hollow cylinder is subjected to convective boundary condition on upper and lower plane surface also zero temperature is maintained on the curved surfaces. The attempt made to analyze the temperature, displacement and thermal stresses of cylinder by using Integral transform technique. Numerical results are computed and represented graphically for the temperature and stress distributions.*

Keywords: *Fractional order theory, hollow cylinder, Hankel and Laplace transform*

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INTRODUCTION

The theory of coupled thermoelasticity is initially introduced by Biot [1] to overcome the first short coming. The governing equations in this coupled theory involved eliminating the first paradox of the classical theory. Due to mixed parabolic hyperbolic type heat equation for the coupled theory the second short coming is shared by both theories, after this many generalizations to the coupled theory were introduced. Important analytical results to the coupled theory were obtained by Hetnarski and Ignaczak [2] and examined several generalizations. A unified generalized thermoelasticity theory was also introduced by Hetnarski and Eslami [3] and also mathematical and mechanical background of thermodynamics, classical thermoelasticity advanced theory and applications, generalized thermoelasticity presented successfully. Ignaczak and Ostoja-Starzeweski [4] explained the mathematical aspects of Lord and Shulman theory [5] illustrated in detail. For a finite propagation speed, generalization of the Fourier law stated by Preziosi [6] and Cattaneo [7].

To find the exact solutions for an isotropic body in various curvilinear coordinate systems for elastic and static thermoelastic problems the Fourier method was used by Podil'chuk and Kirichenko [8] while analytical solutions are available only for a few problems with simple geometries or loading system. Therefore, numerical or experimental analyses are used to solve such problems. In stress analysis of a finite elastic component an analytical/numerical method was developed by Gao and Rowland [9].

For an anisotropic and inhomogeneous material some theorems in the linear theory of thermoelasticity with dual phase-lags was discussed by Kothari and Mukhopadhyay [10]. A three dimensional models of generalized thermoelasticity with one relaxation time and with one and two-temperature determined by Ezzat and Youssef [11], whereas under temperature dependent mechanical properties three dimensional model of the generalized thermoelasticity without energy dissipation was established by Abbas [12]. The establishment of whole theory of fractional derivatives and integrals is done in 19th Century and many existing physical models have been successfully modified by Fractional calculus. Abel [13] gives the first application of fractional derivatives who find the solution of an integral equation by applied fractional calculus in the Tautochrone problem.

The thermoelastic stress, strain, and conductive temperature in a piezo elastic half space body is obtained by Islam and Kanoria [14] for two temperature generalized thermoelasticity theory in the context of the fractional order. Variation of time-fractional differential operators with memory effects was investigated by Povstenko [15, 16]. Time-fractional heat conduction in a composite medium is solved analytically for an infinite matrix and is presented for a spherical inclusion by Povstenko [17]. Associated Thermal Stresses is determined in space with a Source which Varying Harmonically in Time Space in context of Fractional Heat Conduction. Warbhe et al. [18] studied quasi-static approach for fractional-order theory of thermoelasticity in two-dimensional problem of a thin circular plate.

NOTATION AND GOVERNING EQUATIONS

The basic relationship for the problem defined above can be summarized as follows:
The expression for displacement function is defined by [2] as

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \frac{(1+\nu)}{(1-\nu)} a_t\, \theta \tag{1}$$

Where, $\phi = 0$ at $r = a$ and $r = b$ (2)

Here, ν is Poisson's ratio, a_t is coefficient of thermal expansion and θ denotes temperature for length hollow cylinder.

***1.** The Caputo type fractional derivative for nonlocal heat conduction is defined* by [15]

$$\frac{\partial^\alpha f(t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau, \qquad n-1 < \alpha < n \tag{3}$$

To find Laplace transforms of the Caputo derivative it needs to know the initial values of the function $f(t)$ and its integer derivatives of the order $P = 0,1,2,....,n-1$.

$$L\left\{\frac{\partial^\alpha f(t)}{\partial t^\alpha}\right\} = s^\alpha f^*(s) - \sum_{P=0}^{P=n-1} f^{(P)}(0^+) s^{\alpha-1-P}, \qquad n-1 < \alpha < n \tag{4}$$

2. Displacement temperature relationship [2] as:

$$\nabla^2 U - \frac{U}{r^2} + (1+2\nu)^{-1} \frac{\partial e}{\partial r} = 2\frac{(1+\nu)}{(1-\nu)} a_t \frac{\partial \theta}{\partial r} \tag{5}$$

$$\nabla^2 W - \frac{U}{r^2} + (1+2\nu)^{-1} \frac{\partial e}{\partial r} = 2\frac{(1+\nu)}{(1-\nu)} a_t \frac{\partial \theta}{\partial r} \tag{6}$$

Where, e denotes the volume dilation and is defined as $e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z}$.

Here, U and W are radial and axial displacement respectively for uncoupled finite hollow cylinder.

2. Radial and axial displacement U and W *in terms* of thermoelastic displacement function is defined as

$$U = \frac{\partial \phi}{\partial r} \tag{7}$$

$$W = \frac{\partial \phi}{\partial z} \tag{8}$$

3. Stress Displacement relationship is as follows

$$\sigma_r = (\lambda + 2G)\frac{\partial U}{\partial r} + \lambda\left(\frac{U}{r} + \frac{\partial W}{\partial z}\right) \tag{9}$$

$$\sigma_z = (\lambda + 2G)\frac{\partial W}{\partial z} + \lambda\left(\frac{\partial U}{\partial r} + \frac{U}{r}\right) \tag{10}$$

$$\sigma_\theta = (\lambda + 2G)\frac{U}{r} + \lambda\left(\frac{\partial U}{\partial r} + \frac{\partial U}{\partial z}\right) \tag{11}$$

$$\sigma_{rz} = G\left(\frac{\partial W}{\partial r} + \frac{\partial U}{\partial z}\right) \tag{12}$$

Where, $\lambda = \dfrac{2G\nu}{1-2\nu}$ is the Lames' constant, G is the shear modulus.

The stress functions are given by

$$\sigma_{rz}(a, z, t)\big| = 0,\ \sigma_{rz}(b, z, t) = 0,\ \sigma_{rz}(r, z, 0) = 0 \tag{13}$$

$$\sigma_r(r, z, t)\big|_{r=a} = p_1,\ \sigma_r(r, z, t)\big|_{r=b} = -p_0,\ \sigma_z(r, z, t)\big|_{z=0} = 0 \tag{14}$$

Here along the boundaries of the cylinder surface pressures p_1 and p_0 are assumed to be uniform.

FORMULATION OF THE PROBLEM

For unsteady state the heat conduction equation in time fractional order context for finite length hollow cylinder is given as [18]

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} = \frac{1}{K}\frac{\partial^\alpha\theta}{\partial t^\alpha}\ ;\quad a \le r \le b,\quad 0 \le z \le h \tag{15}$$

with initial condition and boundary conditions

$$\theta(r, z, t)\big|_{t=0} = 0 \qquad 0 < \alpha < 2 \tag{16}$$

$$\theta(r, z, t) + C\frac{\partial\theta(r, z, t)}{\partial z}\bigg|_{z=h} = f(r)\,\delta(t) \tag{17}$$

$$\theta(r, z, t) + C\frac{\partial\theta(r, z, t)}{\partial z}\bigg|_{z=0} = g(r)\,\delta(t) \tag{18}$$

$$\theta(r, z, t)\big|_{r=a} = 0 \tag{19}$$

$$\theta(r, z, t)\big|_{r=b} = 0 \tag{20}$$

Equations (1) to (20) constitute the mathematical formulation of the problem under consideration.

SOLUTION OF THE PROBLEM

Applying finite Hankel transform to (15), (16), (17) and (18) and using (19), (20) one obtains

$$\frac{d^2\bar{\theta}}{dz^2} - \lambda_n^2\,\bar{T} = \frac{1}{K}\frac{\partial^\alpha\bar{\theta}}{\partial t^\alpha} \tag{21}$$

$$\bar{\theta}(\lambda_n, z, 0) = 0 \tag{22}$$

$$\bar{\theta}(\lambda_n, h, t) + C\frac{d\bar{\theta}(\lambda_n, h, t)}{dz} = \bar{f}(\lambda_n)\delta(t) \tag{23}$$

$$\bar{\theta}(\lambda_n, 0, t) + C\frac{d\bar{\theta}(\lambda_n, 0, t)}{dz} = \bar{g}(\lambda_n)\delta(t) \tag{24}$$

Where, λ_n is the Hankel transform parameter and $\bar{\theta}$ denotes the finite Hankel transform of θ.

On Applying Laplace transform to the equations (21), (23), (24) and using (22) one obtains

$$\frac{d^2\bar{\theta}^*}{dz^2} - q^2\bar{\theta}^* = 0 \tag{25}$$

Where,

$$q^2 = \lambda_n^2 + \frac{1}{K}\left[s^{\alpha} L\{\theta\} - \sum_{m=0}^{m=n-1} \theta^{(m)}(0^+) s^{\alpha-1-m} \right] \tag{26}$$

$$\bar{\theta}^*(\lambda_n, h, s) + C\frac{d\bar{\theta}^*(\lambda_n, h, s)}{dz} = \bar{f}^*(\lambda_n) \tag{27}$$

$$\bar{\theta}^*(\lambda_n, 0, s) + C\frac{d\bar{\theta}^*(\lambda_n, 0, s)}{dz} = \bar{g}^*(\lambda_n) \tag{28}$$

Where, $\bar{\theta}^*$ denotes the Laplace transform of $\bar{\theta}$ and s is a Laplace transform parameter. The equation (25) is a second order differential equation whose solution is in the form

$$\bar{\theta}^*(\lambda_n, z, s) = A e^{qz} + B e^{-qz} \tag{29}$$

Where, A, B are constants and λ_n are positive roots of the transcendental equation.

$$[I_0(\lambda_n a) K_0(\lambda_n b) - I_0(\lambda_n b) K_0(\lambda_n a)] = 0 \tag{30}$$

Where, I_0 and K_0 are modified Bessel's function of first and second kind of order zero respectively.

Using (27) and (28) in (29) one obtains

$$A = \frac{\bar{f}^*(\lambda_n) - \bar{g}^*(\lambda_n)\ e^{-qh}}{2(1+qC)\sinh(qh)},\ B = -\frac{\bar{f}^*(\lambda_n) - \bar{g}^*(\lambda_n)\ e^{qh}}{2(1-qC)\sinh(qh)}$$

Substituting the values of A and B in (29) one obtains,

$$\begin{aligned}\bar{\theta}^*(\lambda_n, z, s) = {} & \frac{\bar{f}^*(\lambda_n)}{1-q^2C^2}\left(\frac{\sinh(qz)}{\sinh(qh)} - qC\frac{\cosh(qz)}{\sinh(qh)}\right) \\ & - \frac{\bar{g}^*(\lambda_n)}{1-q^2C^2}\left(\frac{\sinh(q(z-h))}{\sinh(qh)} - qC\frac{\cosh(q(z-h))}{\sinh(qh)}\right)\end{aligned} \tag{31}$$

Applying inverse Laplace transform and inverse Hankel transform to (31) one obtains the expression for temperature $\theta(r, z, t)$ as,

$$\theta(r,z,t)=\frac{4K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\ I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-K_0^2(a\lambda_n)}[I_0(r\lambda_n)\ K_0(b\lambda_n)-I_0(b\lambda_n)\ K_0(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi z}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$-\frac{4K\pi(1-C)}{h^2}\frac{\lambda_n^2 I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}[I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right) \tag{32}$$

Here, $E_\alpha(.)$ represents Mittag–Leffler function.

DETERMINATION OF THERMOELASTIC DISPLACEMENT

Using $\theta(r,z,t)$ from (32) in (1) one obtains the thermoelastic displacement function $\phi(r,z,t)$ as,

$$\phi(r,z,t)=\left(\frac{1+\nu}{1-\nu}\right)a_t\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{r^2\lambda_n^2\ I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}[I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi z}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{r^2\lambda_n^2\ I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}[I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right) \tag{33}$$

Radial and axial displacement U and W are obtained by using (33) in (7) and (8) as,

$$U=\left(\frac{1+\nu}{1-\nu}\right)a_t\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\ I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$
$$\times 2r(I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)-r^2\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)k_1(r\lambda_n))]]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi z}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{r^2\lambda_n^2\ I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$
$$\times 2r(I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)-r^2\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]$$
$$\times\frac{\bar{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right) \tag{34}$$

$$W=\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{r^2\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}[I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)$$

$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left(\frac{m\pi}{h}\right)\left[\cos\left(\frac{m\pi z}{h}\right)+\left(\frac{m\pi}{h}\right)\sin\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{r^2\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\left[I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)\right]$$

$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left(\frac{m\pi}{h}\right)\left[\cos\left(\frac{m\pi(z-h)}{h}\right)+\left(\frac{m\pi}{h}\right)\sin\left(\frac{m\pi(z-h)}{h}\right)\right]$$

$$\times\frac{\overline{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right) \qquad (35)$$

DETERMINATION OF THERMOELASTIC DISPLACEMENT

Using (34) and (35) in (9) to (12) the stress function are obtained as,

$$\sigma_r=(\lambda+2G)\left[\left(\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\right]\right.$$

$$\times\,2r(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-4r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))]]$$

$$-\,r^2\lambda_n^2(I_1'(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_1'(r\lambda_n)]$$

$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi z}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$

$$\times\,2r(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)\qquad-4r\lambda_n(I_1(r\lambda_n)I_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$-\,r^2\lambda_n^2(I_1'(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_1'(r\lambda_n)]$$

$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]$$

$$\times\frac{\overline{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$+\,\lambda\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$

$$\times\,2r(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$+\,r^2(I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)]$$

$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]$$

$$+\left(\frac{m\pi}{h}\right)^2\left[\cos\left(\frac{m\pi z}{h}\right)\left(\frac{m\pi}{h}\right)-\sin\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$-\lambda\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$
$$\times\,2r(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$
$$+\,r^2(I_0(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_0(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]$$
$$+\left(\frac{m\pi}{h}\right)^2\left[\cos\left(\frac{m\pi z}{h}\right)\left(\frac{m\pi}{h}\right)-\sin\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}\,E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)\qquad(36)$$

$$\sigma_z=(\lambda+2G)\left[\left(\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\right]\right.$$
$$\times\,[I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)]]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left(\frac{m\pi}{h}\right)^2\left[\cos\left(\frac{m\pi z}{h}\right)\left(\frac{m\pi}{h}\right)-\sin\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}\,E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\;\times[I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)]]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left(\frac{m\pi}{h}\right)^2\left[\cos\left(\frac{m\pi(z-h)}{h}\right)\left(\frac{m\pi}{h}\right)-\sin\left(\frac{m\pi(z-h)}{h}\right)\right]$$
$$\times\,\frac{\bar{g}(\lambda_n)}{1-\lambda_n^2C^2}\,E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$+\lambda\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$
$$\times\,4(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-5r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$
$$-\,r^2(I_1'(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1'(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}\,E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$-\lambda\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$
$$\times\,4(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-5r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$
$$-\,r^2(I_1'(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1'(r\lambda_n)]$$
$$\times\sum_{m=1}^{\infty}m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\times\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}\,E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$
$$(37)$$

$$\sigma_\theta=(\lambda+2G)\left[\left(\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\right]\right.$$
$$\times\,2(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))]]$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left[\sin\left(\frac{m\pi z}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi z}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-K_0^2(a\lambda_n)}$$

$$\times\,2(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{g}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$+\lambda\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$

$$\times\,2(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-4r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$-\,r^2\lambda_n^2(I_1'(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1'\;(r\lambda_n)]\sum_{m=1}^{\infty} m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]$$

$$\times\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$

$$\times\,2(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-4r\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$-\,r^2\lambda_n^2(I_1'(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1'\;(r\lambda_n)]$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left[\sin\left(\frac{m\pi(z-h)}{h}\right)-\left(\frac{m\pi}{h}\right)\cos\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$+\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{r^2\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\,(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left(\frac{m\pi}{h}\right)^2\left[\cos\left(\frac{m\pi(z-h)}{h}\right)\left(\frac{m\pi}{h}\right)-\sin\left(\frac{m\pi(z-h)}{h}\right)\right]\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)\tag{38}$$

$$\sigma_{rz}=2G\left[\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{n=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}\right]$$

$$\times\,2(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)-r^2\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))]]$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left(\frac{m\pi}{h}\right)\left[\cos\left(\frac{m\pi z}{h}\right)+\left(\frac{m\pi}{h}\right)\sin\left(\frac{m\pi z}{h}\right)\right]$$

$$\frac{\bar{f}(\lambda_n)}{1-\lambda_n^2C^2}E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha-t'^\alpha)\right)$$

$$-\left(\frac{1+\nu}{1-\nu}\right)a_t\,\frac{K\pi(1-C)}{h^2}\sum_{h=1}^{\infty}\frac{\lambda_n^2\,I_0^2(\lambda_n a)}{I_0^2(b\lambda_n)-I_0^2(a\lambda_n)}$$

$$\times[\,2r(I_0(r\lambda_n)\,K_0(b\lambda_n)-I_0(b\lambda_n)\,K_0(r\lambda_n)\,]-r^2\lambda_n(I_1(r\lambda_n)K_0(b\lambda_n)-I_0(b\lambda_n)K_1(r\lambda_n))$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\left(\frac{m\pi}{h}\right)\left[\cos\left(\frac{m\pi(z-h)}{h}\right)+\left(\frac{m\pi}{h}\right)\sin\left(\frac{m\pi(z-h)}{h}\right)\right]$$

$$\times \frac{\bar{g}(\lambda_n)}{1-\lambda_n^2 C^2} E_\alpha\left(-K\left(\lambda_n^2+\frac{m^2\pi^2}{h^2}\right)(t^\alpha - t'^\alpha)\right) \tag{39}$$

SPECIAL CASE AND NUMERICAL RESULTS

Numerical computations for temperature and stresses has been done by using MATHEMATICA software for different value of α and shown graphically in Figures 1–5. The material properties of copper have been considered for the numerical computations.

$E = 120\text{GPa},\ \nu = 0.34,\ \rho = 8.954\times10^3\ \text{Kg m}^{-3},\ K = 0.386\times10^3\ \text{Wm}^{-1}\text{K}^{-1},$

Set $f(r,t) = (1-e^{-t})\left(\frac{\sin(r-a)\sin(b-r)}{r}\right)e^h$ and $g(r,t) = (1-e^{-t})\left(\frac{\sin(r-a)\sin(b-r)}{r}\right)$

$\alpha = \frac{2K\pi}{ah^2}[h^2+2h][\cos(a+b)-\cos(3b-a)]$, $k = 0.86$, $t = 1$ sec., $a = 1$, $b = 2$ and $h = 1$ in (32) to obtain

$$\frac{\theta(r,z,t)}{\alpha} = \sum_{n=1}^{\infty}\frac{I_0^2(\lambda_n)}{I_0^2(2\lambda_n)-I_0^2(\lambda_n)}[I_0(r\lambda_n)K_0(2\lambda_n)-I_0(2\lambda_n)K_0(r\lambda_n)]$$

$$\times\sum_{m=1}^{\infty} m(-1)^{m+1}\{[\sin(3.14mz)-(3.14m)\cos(3.14mz)]e$$

$$-[\sin(3.14m(z-1)-(3.14m)\cos(3.14m(z-1))]\}\times\int_0^1\frac{(1-e^{-t'})}{1-\lambda_n^2c^2}e^{-0.86(\lambda_n^2+9.6m^2)(1-t')}dt' \tag{40}$$

GRAPHICAL ANALYSIS

Figure 1 shows the variation of temperature along the radial direction for different value of α. The graphical plot indicates that the temperature goes on increases as going radially outward direction.

Figure 2 shows the variation of the radial stress along the radial direction. Initially σ_r is at zero and as going radially outward σ_r starts decreasing for different value of α.

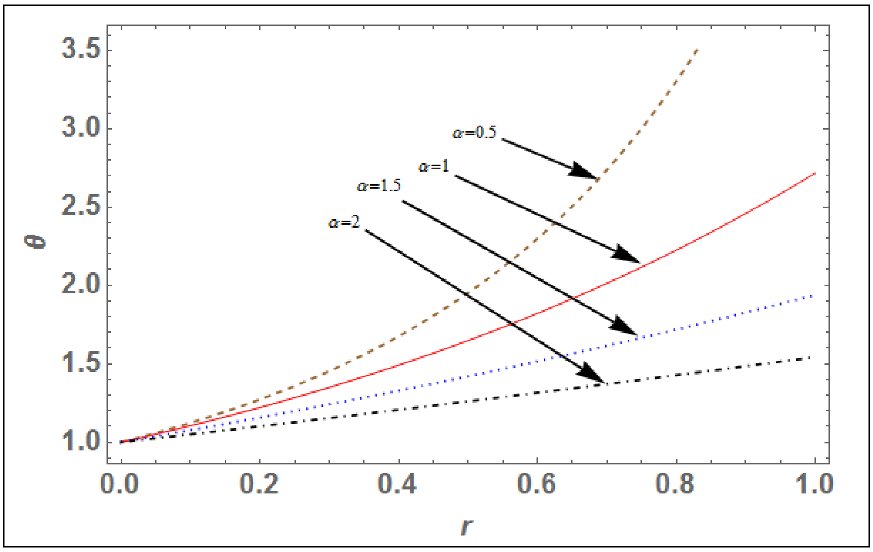

***Fig. 1:** Temperature θ with Respect to Radius r for Different Values of α.*

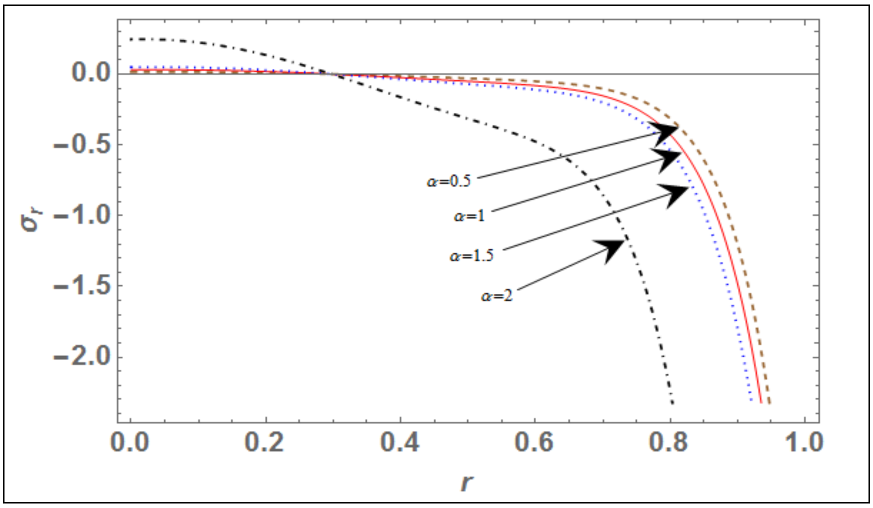

Fig. 2: *Stress* σ_r *with Respect to Radius* r *for Different Values of* α .

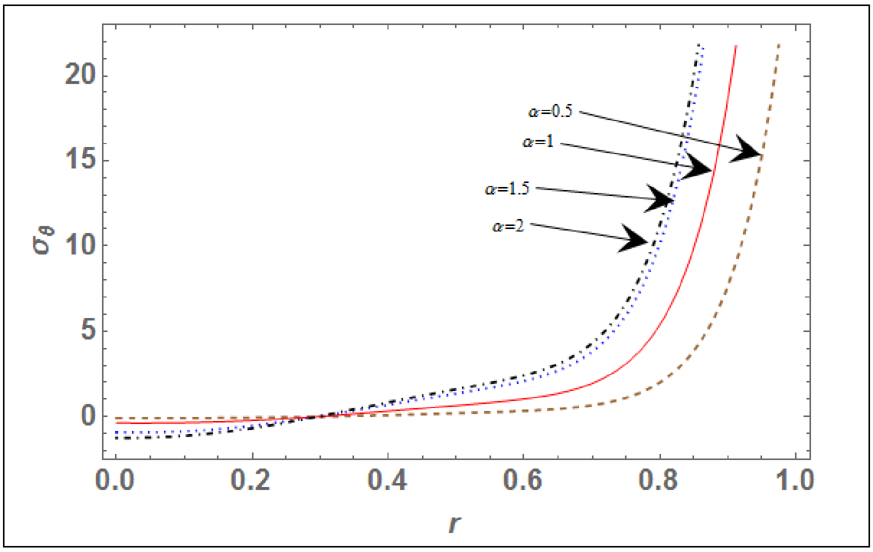

Fig. 3: *Stress* σ_θ *with Respect to Radius* r *for Different Values of* α .

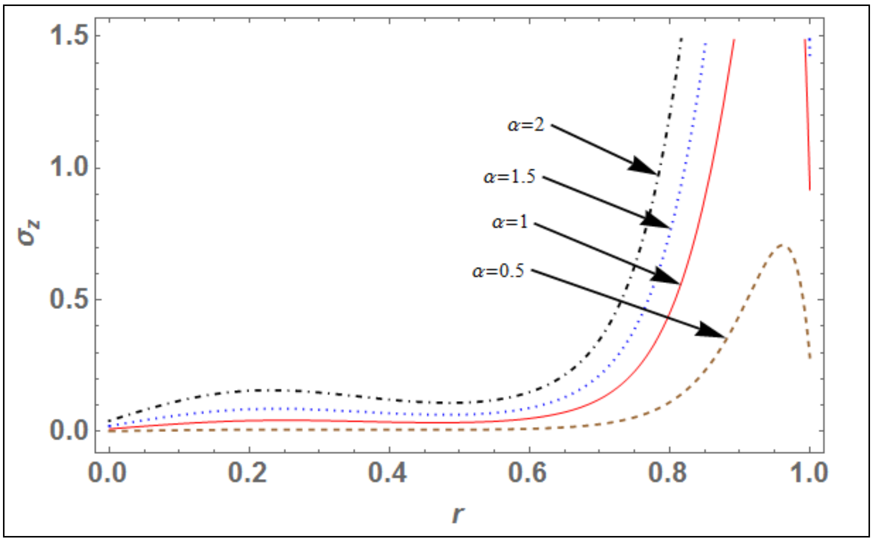

Fig. 4: *Stress* σ_z *with Respect to Radius* r *for Different Values of* α .

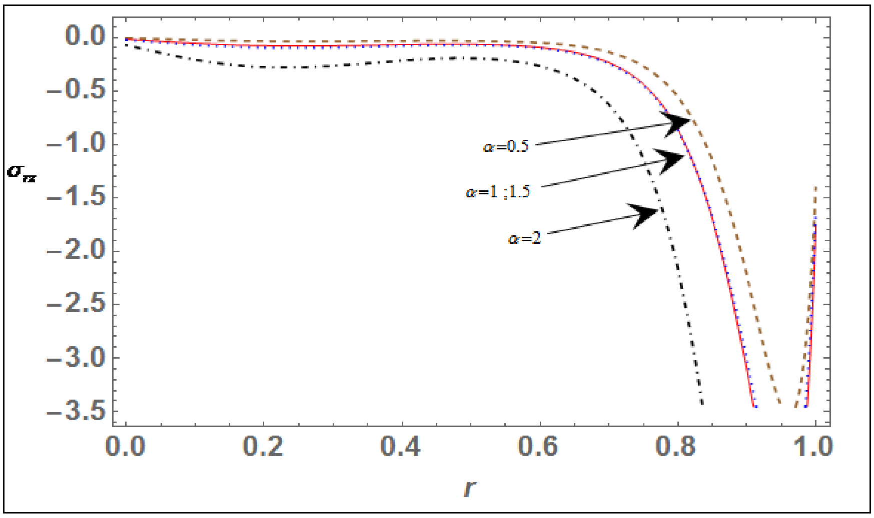

Fig. 5: *Stress σ_{rz} with Respect to Radius r for Different Values of α.*

Figure 3 represents the variation of stresses σ_θ along the radial direction for different value of α. It is observed that stress function σ_θ goes on increasing in radial outward direction.

Figure 4 represents the variation of stresses σ_z along the radial direction for different value of α. It is observed that stress function σ_z smoothly varies from $0 \le r \le 0.6$ and after these increases instantly in range $0.6 \le r \le 1$ along radial outward direction.

Figure 5 represents the variation of stresses σ_{rz} along the radial direction for different value of α. It is observed that stress function σ_{rz} smoothly varies from $0 \le r \le 0.6$ and after these decreases instantly in range $0.6 \le r \le 1$ along radial outward direction.

CONCLUSIONS

In this work we discussed fractional order theory of thermoelasticity for the finite isotropic hollow cylinder under convective boundary condition on upper and lower plane surface. Here we analyzed only quasi-static uncoupled approach for hollow cylinder without considering the inertia term in the basic equation of motion. The associated thermal stresses are obtained by using the displacement potential function with the use of Integral transform technique. The infinite wave propagation in terms of heat energy is observed due to the parabolic nature of heat conduction equation. Numerical results are computed and represented graphically for the temperature and stress distributions.

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