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The Relation between Double Laplace Transform and Double Hankel Transform with Applications II

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ABSTRACT

The object of this paper is to establish a relationship between the double Laplace transform and the double Hankel transform. A double Laplace-Hankel transform of the product of H-functions of one and two variables is then obtained. Application of our main result, summation formula and some interesting special cases have also been discussed.

Keywords: *Double Laplace transform, double Hankel transform, Mellin-Bernes contour integral, H-function*

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1. INTRODUCTION

If $F(p_1, p_2)$ is the double Laplace transform of $f(x, y)$, then

$$F(p_1, p_2) = \int_0^\infty \int_0^\infty e^{-p_1 x - p_2 y} f(x, y) dxdy; \quad \mathrm{Re}(p_1) > 0, \mathrm{Re}(p_2) > 0 \tag{1}$$

If $H(p_1, p_2)$ is the Double Hankel transform of $f(x, y)$, then

$$H(p_1, p_2) = \int_0^\infty \int_0^\infty xJ_\nu(p_1 x) yJ_\mu(p_2 y) f(x, y) dxdy \tag{2}$$

Provided that $xJ_\nu(p_1 x) > 0, yJ_\mu(p_2 y) > 0$.

The following formula is required in the proof:

$$\int_0^\infty \int_0^\infty x^{s-1} y^{t-1} H\left[ax^\lambda, by^\mu\right] dxdy = \frac{a^{-s/\lambda} b^{-t/\mu}}{\lambda\mu} \phi\left(-\frac{s}{\lambda}, -\frac{t}{\mu}\right) \theta_2\left(-\frac{s}{\lambda}\right) \theta_3\left(-\frac{t}{\mu}\right) \tag{3}$$

H[x] represents the *H*-function of Fox [1].

The H -function of two variables [2] using the following notation, which is due essentially to Srivastava and Panda [3] is defined and represented as:

$$H[x, y] = H\begin{bmatrix} x \\ y \end{bmatrix} = H^{0,n_1:m_2,n_2:m_3,n_3}_{p_1,q_1:p_2,q_2:p_3,q_3}\left[\begin{matrix} x \\ y \end{matrix}\middle| \begin{matrix} (a_j;\alpha_j,A_j)_{1,p_1}:(c_j,\gamma_j)_{1,p_2},(e_j,E_j)_{1,p_3} \\ (b_j;\beta_j,B_j)_{1,q_1}:(d_j,\delta_j)_{1,q_2},(f_j,F_j)_{1,q_3} \end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi(\xi,\eta)\,\theta_2(\xi)\,\theta_3(\eta)\,x^{\xi}y^{\eta}\,d\xi\,d\eta \tag{4}$$

where,

$$\phi(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\alpha_j\xi-A_j\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\beta_j\xi+B_j\eta)} \tag{5}$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{n_2}\Gamma(1-c_j+\gamma_j\xi)\prod_{j=1}^{m_2}\Gamma(d_j-\delta_j\xi)}{\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\gamma_j\xi)\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\delta_j\xi)} \tag{6}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{n_3}\Gamma(1-e_j+E_j\eta)\prod_{j=1}^{m_3}\Gamma(f_j-F_j\eta)}{\prod_{j=n_3+1}^{p_3}\Gamma(e_j-E_j\eta)\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+F_j\eta)} \tag{7}$$

2. MAIN RESULT

Theorem: If $F(p_1, p_2)$ is the Laplace transform, then

$$F(p_1,p_2) = \frac{(p_1)^{-\nu-s_1}(-p_2)^{-\mu-s_2}}{2^{-\nu-\mu-2(s_1+s_2)}}\Gamma(\nu+s_1+1)\Gamma(\mu+s_2+1)$$
$$\int_0^{\infty}\int_0^{\infty} t_1^{-s_1-\nu}t_2^{-s_2-\mu}J_{\nu}(p_1t_1)J_{\mu}(p_2t_2)f(t_1,t_2)\,dt_1\,dt_2 \tag{8}$$

Proof: From Eq. (1), $F(p_1,p_2) = \int_0^{\infty}\int_0^{\infty} e^{-p_1t_1-p_2t_2}f(t_1,t_2)\,dt_1\,dt_2$

$$= \sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\frac{(-p_1)^{s_1}}{s_1!}\frac{(-p_2)^{s_2}}{s_2!}\int_0^{\infty}\int_0^{\infty} t_1^{s_1}t_2^{s_2}f(t_1,t_2)\,dt_1\,dt_2$$

$$=\frac{(p_1)^{-\nu-s_1}}{2^{-\nu-2s_1}}\frac{(p_2)^{-\mu-s_2}}{2^{-\mu-2s_2}}\Gamma(\nu+s_1+1)\Gamma(\nu+s_2+1)$$

$$\int_0^\infty\int_0^\infty t_1^{-\nu-s_1}t_2^{-\mu-s_2}\sum_{s_1=0}^{\infty}\frac{(-1)^{s_1}}{s_1!\Gamma(\nu+s_1+1)}\left(\frac{p_1t_1}{2}\right)^{\nu+2s_1}\sum_{s_2=0}^{\infty}\frac{(-1)^{s_2}}{s_2!\Gamma(\nu+s_2+1)}\left(\frac{p_2t_2}{2}\right)^{\mu+2s_2}f(t_1,t_2)\,dt_1\,dt_2$$

$$=\frac{(p_1)^{-\nu-s_1}(-p_2)^{-\mu-s_2}}{2^{-\nu-\mu-2(s_1+s_2)}}\Gamma(\nu+s_1+1)\Gamma(\mu+s_2+1)\int_0^\infty\int_0^\infty t_1^{-s_1-\nu}t_2^{-s_2-\mu}J_\nu(p_1t_1)J_\mu(p_2t_2)f(t_1,t_2)\,dt_1\,dt_2$$

3. A DOUBLE HANKEL TRANSFORM

$$\int_0^\infty\int_0^\infty xJ_\nu(p_1x)yJ_\mu(p_2y)H_{p,q}^{m,n}\left[cx^\lambda y^\delta\left|\begin{matrix}{}_1(g_j,G_j)_p\\ {}_1(h_j,H_j)_q\end{matrix}\right.\right]H\left[ax^\gamma,by^\eta\right]dxdy$$

$$=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}p_1^{\nu+2r_1}p_2^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)/\gamma}b^{-(\mu+2r_2+2)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1\left(1-d_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j\right)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1\left(1-c_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j\right)_{n_2}\end{matrix}\right.\right.$$

$$\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\\ {}_1\left(1-e_j-\left(\frac{\mu+2r_2+2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\end{matrix}$$

$$\left.\begin{matrix}{}_1\left(1-b_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+2}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\ {}_1\left(1-a_j-\left(\frac{\nu+2r_1+2}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+2}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right] \tag{9}$$

provided,

$$\lambda,\delta>0;\eta>0;|\arg c|<\frac{1}{2}\Delta\pi,\Delta>0$$

where,

$$\Delta=\sum_{j=1}^{m}H_j-\sum_{j=m+1}^{q}H_j+\sum_{j=1}^{n}G_j-\sum_{j=n+1}^{p}G_j$$

$$\mathrm{Re}[(\nu+2r_1+2+\gamma(d_i/D_i)+\lambda(h_j/H_j)]>0;i=1,\dots,m_2;\ j=1,\dots,m$$

$$\mathrm{Re}[(\mu+2r_2+2+\eta\left(f_i/F_i\right)+\delta\left(h_j/H_j\right)]>0; i=1,\ldots,m_3; j=1,\ldots,m$$

$$\mathrm{Re}\left[\nu+2r_1+2-\gamma\left(\frac{1-c_i}{C_i}\right)-\lambda\left(\frac{1-g_j}{G_j}\right)\right]<0; i=1,..,n_2; j=1,\ldots,n$$

Proof: To prove Eq. (9), expand Bessel function in series form and substitute the Mellin-Bernes contour integral for $H[cx^{\lambda}y^{\delta}]$ on the left hand side, then interchange the order of contour integral and the (x,y)-integrals. Finally, we arrive at our result on evaluating the (x,y) integral by using the result in Eq. (3).

4. APPLICATION OF THE MAIN RESULT

If $f(t_1,t_2)=t_1^{\nu+2r_1+2}t_2^{\mu+2r_2+2}J_{\nu}(p_1t_1)J_{\mu}(p_2t_2)H_{p,q}^{m,n}\left[ct_1^{\lambda}t_2^{\delta}\,\middle|\,\begin{matrix}{}_1(g_j,G_j)_p\\ {}_1(h_j,H_j)_q\end{matrix}\right]H\left[at^{\gamma},bt^{\eta}\right]$;

Eq. (8) becomes the double Laplace-Hankel transform of the product of H-functions of one and two variables and takes the following form:

$$\int_0^{\infty}\int_0^{\infty}e^{-P_1t_1-P_2t_2}t_1^{\nu+2r_1+2}t_2^{\mu+2r_2+2}J_{\nu}(p_1t_1)J_{\mu}(p_2t_2)H_{p,q}^{m,n}\left[ct_1^{\lambda}t_2^{\delta}\,\middle|\,\begin{matrix}{}_1(g_j,G_j)_p\\ {}_1(h_j,H_j)_q\end{matrix}\right]H\left[at_1^{\gamma},bt_2^{\eta}\right]dt_1dt_2=$$

$$\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{s_1}(-1)^{s_2}(-1)^{r_1}(-1)^{r_2}p_1^{\nu+s_1+2r_1}p_2^{\mu+s_2+2r_2}}{s_1!s_2!r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)/\gamma}b^{-(\mu+2r_2+2)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\,\middle|\,\begin{matrix}{}_1(g_j,G_j)_n,{}_1(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j)_{n_2}\end{matrix}\right.$$

$$\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\\ {}_1\left(1-e_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\end{matrix}$$

$$\left.\begin{matrix}{}_1\left(1-b_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\ {}_1\left(1-a_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right] \quad (10)$$

Provided the conditions are same as that of Eq. (9) with $\mathrm{Re}(p_1)>0$, $\mathrm{Re}(p_2)>0$.

Proof: In Eq. (8)put

$$f(t_1,t_2)=t_1^{\nu+2r_1+2}t_2^{\mu+2r_2+2}J_\nu(p_1t_1)J_\mu(p_2t_2)H_{p,q}^{m,n}\left[ct_1^\lambda t_2^\delta\left|\begin{matrix}{}_1(g_j,G_j)_p\\{}_1(h_j,H_j)_q\end{matrix}\right.\right]H\left[at^\gamma bt^\eta\right]$$

and use Eq. (9) to get,

$$H(p_1,p_2)=\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}p_1^{\nu+2r_1}p_2^{\mu+2r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+1)/\gamma}b^{-(\mu+2r_2+1)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1\left(1-d_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j\right)_{m_2}\\{}_1(h_j,H_j)_m,{}_1\left(1-c_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j\right)_{n_2}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+1}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\;{}_1\left(1-b_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+1}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\{}_1\left(1-e_j-\left(\frac{\mu+2r_2+1}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\;{}_1\left(1-a_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+1}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right]$$

Hence F(p_1 ,p_2) = the right hand side of Eq. (10).

5. SUMMATION FORMULA

$$\int_0^\infty\int_0^\infty e^{-p_1t_1-p_2t_2}J_\nu(p_1t_1)J_\mu(p_2t_2)H_{p,q}^{m,n}\left[ct_1^\lambda t_2^\delta\left|\begin{matrix}{}_1(g_j,G_j)_p\\{}_1(h_j,H_j)_q\end{matrix}\right.\right]H\left[at_1^\gamma,bt_2^\eta\right]dt_1dt_2=$$

$$\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{s_1}(-1)^{s_2}(-1)^{r_1}(-1)^{r_2}p_1^{\nu+s_1+2r_1}p_2^{\mu+s_2+2r_2}}{s_1!s_2!r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)/\gamma}b^{-(\mu+2r_2+2)/\eta}(\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j\right)_{m_2}\\{}_1(h_j,H_j)_m,{}_1\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j\right)_{n_2}\end{matrix}\right.\right.$$

$$\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\\{}_1\left(1-e_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\end{matrix}$$

$$\left.\begin{matrix}{}_1\left(1-b_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+1}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\{}_1\left(1-a_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+1}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right] \tag{11}$$

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Evaluating the left hand side of Eq. (11), using [4, 5 Eq.(8.5.6),p.150], the following summation formula is obtained:

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}\,p_1^{-2}p_2^{-2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-\xi}b^{-\eta}$$

$$H^{m+n_2+n_3,n+2+m_2+m_3}_{p+2+q_1+q_2+q_3,q+p_1+p_2+p_3}\left[cp_1^{-\delta}p_2^{-\eta}ab\left|\begin{matrix}{}_1(g_j,G_j)_n,(-\nu-2r_1,\lambda),(-\mu-2r_2,\delta),{}_1(1-d_j;\delta D_j)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1(1-c_j;\gamma C_j)_{n_2}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}{}_1\left(1-f_j,\eta F_j\right)_{q_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j,\delta D_j\right)_{q_2}\quad{}_1\left(1-b_j+\xi\beta_j+\eta B_j\right)_{q_1}\\ {}_1\left(1-e_j,\eta E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j,\gamma C_j\right)_{p_2}\quad{}_1\left(1-a_j+\xi\alpha_j+\eta A_j\right)_{p_1}\end{matrix}\right]$$

$$=$$

$$\sum_{s_1=0}^{\infty}\sum_{s_2=0}^{\infty}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{s_1}(-1)^{s_2}(-1)^{r_1}(-1)^{r_2}P_1^{\nu+s_1+2r_1}P_2^{\mu+s_2+2r_2}}{s_1!s_2!r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)/\gamma}b^{-(\mu+2r_2+2)/\eta}(\gamma\eta)^{-1}$$

$$H^{m+n_2+n_3,n+m_2+m_3}_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j;\frac{\lambda}{\gamma}D_j)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j;\frac{\lambda}{\gamma}C_j)_{n_2}\end{matrix}\right.\right.$$

$$\left.\begin{matrix}{}_1\left(1-f_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)F_j,\frac{\delta}{\eta}F_j\right)_{p_3},{}_{n+1}(g_j,G_j)_p,{}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j,\frac{\lambda}{\gamma}D_j\right)_{q_2}\quad{}_1\left(1-b_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\beta_j-\left(\frac{\mu+2r_2+1}{\eta}\right)B_j+\frac{\lambda}{\gamma}\beta_j+\frac{\delta}{\eta}B_j\right)_{q_1}\\ {}_1\left(1-e_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)E_j,\frac{\delta}{\eta}E_j\right)_{p_3},{}_{m+1}(h_j,H_j)_q,{}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j,\frac{\lambda}{\gamma}C_j\right)_{p_2}\quad{}_1\left(1-a_j-\left(\frac{\nu+2r_1+1}{\gamma}\right)\alpha_j-\left(\frac{\mu+2r_2+1}{\eta}\right)A_j+\frac{\lambda}{\gamma}\alpha_j+\frac{\delta}{\eta}A_j\right)_{p_1}\end{matrix}\right]$$

Provided the conditions are same as that of Eq. (9) with $\mathrm{Re}(p_1) > 0$, $\mathrm{Re}(p_2) > 0$.

6. SPECIAL CASES

(1) In Eq. (10) let $P_1 \to 0$, $P_2 \to 0$ and $\gamma = \eta = 1$, to get the following result:

$$\int_0^{\infty}\int_0^{\infty}t_1^{\nu+2r_1+2}t_2^{\mu+2r_2+2}J_\nu(P_1t_1)J_\mu(P_2t_2)H^{m,n}_{p,q}\left[ct_1^{\lambda}t_2^{\delta}\left|\begin{matrix}{}_1(g_j,G_j)_p\\ {}_1(h_j,H_j)_q\end{matrix}\right.\right]H\left[at_1,bt_2\right]dt_1dt_2=$$

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(-1)^{r_1}(-1)^{r_2}}{r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}}a^{-(\nu+2r_1+2)}b^{-(\mu+2r_2+2)}$$

$$H^{m+n_2+n_3,n+m_2+m_3}_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}\left[ca^{-\lambda}b^{-\delta}\left|\begin{matrix}{}_1(g_j,G_j)_n,{}_1(1-d_j-(\nu+2r_1+2)D_j;\lambda D_j)_{m_2}\\ {}_1(h_j,H_j)_m,{}_1(1-c_j-(\nu+2r_1+2)C_j;\lambda C_j)_{n_2}\end{matrix}\right.\right.$$

$$\left. \begin{matrix} {}_1\left(1-f_j-(\mu+2r_2+2)F_j,\delta F_j\right)_{p_3}, {}_{n+1}(g_j,G_j)_p, {}_{n_2+1}\left(1-d_j-(\nu+2r_1+2)D_j,\lambda D_j\right)_{q_2} \; {}_1\left(1-b_j-(\nu+2r_1+2)\beta_j-(\mu+2r_2+2)B_j+\lambda\beta_j+\delta B_j\right)_{q_1} \\ {}_1\left(1-e_j-(\mu+2r_2+2)E_j,\delta E_j\right)_{p_3}, {}_{m+1}(h_j,H_j)_q, {}_{n_2+1}\left(1-c_j-(\nu+2r_1+2)C_j,\lambda C_j\right)_{p_2} \; {}_1\left(1-a_j-(\nu+2r_1+2)\alpha_j-(\mu+2r_2+2)A_j+\lambda\alpha_j+\delta A_j\right)_{p_1} \end{matrix} \right] \tag{12}$$

(2) In Eq. (10) take $n_1 = p_1 = q_1 = 0$, to get the double Laplace-Hankel transform of the product of three single *H*-functions of Fox as:

$$\int_0^\infty\int_0^\infty e^{-p_1t_1-p_2t_2} t_1^{\nu+2r_1+2} t_2^{\mu+2r_2+2} J_\nu(p_1t_1) J_\mu(p_2t_2) H_{p,q}^{m,n}\left[ct_1^\lambda t_2^\delta \middle| \begin{matrix} {}_1(g_j,G_j)_p \\ {}_1(h_j,H_j)_q \end{matrix}\right]$$

$$H_{p_2,q_2}^{m_2,n_2}\left[ct_1^\gamma \middle| \begin{matrix} {}_1(c_j,C_j)_{p_2} \\ {}_1(d_j,D_j)_{q_2} \end{matrix}\right] H_{p_3,q_3}^{m_3,n_3}\left[ct_2^\eta \middle| \begin{matrix} {}_1(e_j,E_j)_{p_3} \\ {}_1(f_j,F_j)_{q_3} \end{matrix}\right] dt_1 dt_2 =$$

$$\sum_{s_1=0}^\infty\sum_{s_2=0}^\infty\sum_{r_1=0}^\infty\sum_{r_2=0}^\infty \frac{(-1)^{s_1}(-1)^{s_2}(-1)^{r_1}(-1)^{r_2} p_1^{\nu+s_1+2r_1} p_2^{\mu+s_2+2r_2}}{s_1!s_2!r_1!r_2!\Gamma(\nu+r_1+1)\Gamma(\mu+r_2+1)\,2^{\nu+\mu+2(r_1+r_2)}} a^{-(\nu+2r_1+2)/\gamma} b^{-(\mu+2r_2+2)/\eta} (\gamma\eta)^{-1}$$

$$H_{p+q_1+q_2+q_3,q+p_1+p_2+p_3}^{m+n_2+n_3,n+m_2+m_3}\left[ca^{-\lambda/\gamma}b^{-\delta/\eta} \middle| \begin{matrix} {}_1(g_j,G_j)_n, {}_1\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j; \frac{\lambda}{\gamma}D_j\right)_{m_2} \\ {}_1(h_j,H_j)_m, {}_1\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j; \frac{\lambda}{\gamma}C_j\right)_{n_2} \end{matrix}\right.$$

$$\left. \begin{matrix} {}_1\left(1-f_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)F_j, \frac{\delta}{\eta}F_j\right)_{p_3}, {}_{n+1}(g_j,G_j)_p, {}_{n_2+1}\left(1-d_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)D_j, \frac{\lambda}{\gamma}D_j\right)_{q_2} \\ {}_1\left(1-e_j-\left(\frac{\mu+2r_2+s_2+2}{\eta}\right)E_j, \frac{\delta}{\eta}E_j\right)_{p_3}, {}_{m+1}(h_j,H_j)_q, {}_{n_2+1}\left(1-c_j-\left(\frac{\nu+2r_1+s_1+2}{\gamma}\right)C_j, \frac{\lambda}{\gamma}C_j\right)_{p_2} \end{matrix} \right] \tag{13}$$

Provided the conditions are same as that of Eq. (9) with $p_1 = q_1 = 0$; $Re(p_1) > 0$, $Re(p_2) > 0$.

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